# Computing limit linear series with infinitesimal methods

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## Abstract:

Alexander and Hirschowitz [1] determined the Hilbert function of a generic union of fat points in a projective space when the number of fat points is much bigger than the greatest multiplicity of the fat points. Their method is based on a lemma which determines the limit of a linear system depending on fat points which approach a divisor.
On the other hand, Nagata [10], in connection with its counter example to the fourteenth problem of Hilbert determined the Hilbert function $H(d)$ of the union of $k^2$ points of the same multiplicity $m$ in the plane up to degree $d = km$.
We introduce a new method to determine limits of linear systems. This generalizes the result by Alexander and Hirschowitz. Our main application of this method is the conclusion of the work initiated by Nagata: we compute $H(d)$ for all $d$. As a second application, we determine the generic successive collision of four fat points of the same multiplicity in the plane.

## 1 Introduction

Let $X$ be a (quasi-)projective scheme, $\mathcal{L}$ a linear system on $X$ and $Z \subset X$ a generic 0-dimensional subscheme. In this paper, we adress the problem of determining the dimension of $\mathcal{L}(-Z)$, or more precisely the limit of $\mathcal{L}(-Z)$ when $Z$ specializes to a subscheme $Z'$.
Our result gives an estimate of this limit when $Z$ moves to a divisor and satisfies suitable conditions( $Z$ is the generic embedding of a union $Z_1 \cup Z_2 \cdots \cup Z_s$ of monomial schemes). More precisely, we introduce a combinatorical procedure to construct a system $\mathcal{L}'$, "simpler" than $\mathcal{L}$ in the sense that it has smaller degree, and we settle an inclusion $\lim \mathcal{L}(-Z) \subset \mathcal{L}'$. In concrete exemples (see the applications below), the inclusion suffices to compute $\dim \mathcal{L}(-Z)$: there is an expected dimension $d_e$ which verifies

$$d_e \leq \dim \mathcal{L}(-Z) = \dim \lim \mathcal{L}(-Z) \leq \dim \mathcal{L}' = d_e,$$

hence $\dim \mathcal{L}(-Z) = d_e$.
To give a flavour of the theorem, suppose for simplicity that $Z$ is the generic

fiber of a subscheme $F \subset X \times \mathbb{A}^1$ flat over $\mathbb{A}^1 = Spec\ k[t]$ and such that the support of the fiber $F(t)$ approaches a divisor $D$ when $t \to 0$. We find an integer $r$ and a residual scheme $F_{res} \subset F(0)$ such that

$$\lim_{t \to 0} \mathcal{L}(-F(t)) \subset \mathcal{L}(-rD - Z_{res}).$$

There is a trivial inclusion

$$\lim_{t \to 0} \mathcal{L}(-F(t)) \subset \mathcal{L}(-F(0)),$$

but of course our result is more detailed and is not reductible to this trivial case. In the examples we consider, the last inclusion of the tower

$$\lim_{t \to 0} \mathcal{L}(-F(t)) \subset \mathcal{L}(-rD - Z_{res}(0)) \subset \mathcal{L}(-F(0))$$

is always a strict inclusion.
The method to prove the result is infinitesimal in nature. There is a unique flat family $G$ over $\mathbb{A}^1$ whose fiber over a general $t \neq 0$ is $\mathcal{L}(-F(t))$. Our theorem is obtained with a careful analysis of the restrictions $G \times_{\mathbb{A}^1} Spec\ k[t]/(t^{n_i}) \subset G$ for well chosen integers $n_1, \dots, n_r$.
Our theorem generalizes the main lemma of Alexander-Hirschowitz [1]. Their statement corresponds essentially to ours in the special case $r = 1$. However, the proofs are different. In fact, when Alexander-Hirschowitz published their theorem, our theorem did already exist in a weaker version where the 0-dimensional subscheme $Z$ moving to the divisor had to be supported by a unique point. The current version is a merge which contains both our earlier version and Alexander-Hirschowitz version.

As an application of our theorem, we extend results by Nagata relative to the Hilbert functions of fat points in the plane. In connection with his construction of the counter example to the fourteenth problem of Hilbert, Nagata proved that the Hilbert function of a generic union $Z$ of $k^2$ fat points of the same multiplicity $m$ in $\mathbb{P}^2$ is $H_Z(d) = \frac{(d+1)(d+2)}{2}$ if the degree is not to big, namely if $d \leq km$. This result is asymptotically optimal in $m$ in the sense that it is sufficient to compute the Hilbert function up to the critical degree $d = km + [\frac{k}{2}]$ to determine the whole Hilbert function. Nagata was just missing the last extreme hardest $[\frac{k}{2}]$ cases. We compute the Hilbert function for every degree: $H_Z(d) = min(\frac{(d+1)(d+2)}{2}, k^2\frac{m(m+1)}{2})$. This result was already proved when the number of points is a power of four in [8] by methods relying on the geometry of integrally closed ideals which we could not push further.
Putting the result in perspective, we recall that a consequence of Alexander-Hirschowitz [1] is that the Hilbert function of a generic union of $k$ fat points in the plane of multiplicity $m_1, \dots, m_k$ is $H_Z(d) = min(\frac{(d+1)(d+2)}{2}, \sum_{i=1}^{k} \frac{m_i(m_i+1)}{2})$ provided $k >> max(m_i)$. In view of their result, we are left with the cases when the multiplicities are not too small with respect to the number of points. Among these, it is known empirically that the hardest cases are those with a fixed number of points and big multiplicities. Our theorem includes such cases.

As a second application, we compute the generic successive collision of four fat points in the plane of the same multiplicity (recall that a *successive* collision of punctual schemes $Z_1, \dots, Z_s$ is a subscheme obtained as a flat limit when the $Z_i$'s approach one after the other, ie. you first collide $Z_1$ and $Z_2$ in a subscheme $Z_{12}$, then you collide $Z_3$ with the previous collision $Z_{12}$ and so on... A *generic* successive collision is a successive collision where by definition the $Z_i$'s move on generic curves of high degree ).
Let us explain the motivations for such a computation. First, collisions determine the Hilbert function of the generic union $Z$ of the fat points. Indeed, there exist "universal" collisions $C_0$ on which one can read off the Hilbert function of $Z$: $\forall d,\ H_Z(d) = H_{C_0}(d)$ [4]. Moreover, constructing collisions is a useful technical tool of the Horace method (see [7]).
However, determining all collisions of any number of fat points is far beyond our knowledge since this problem is far more difficult than the open and long standed problem of determining the Hilbert function of a generic union of fat points. It is thus natural to restrict our attention to special collisions. In view of the postulation problem, one looks for collisions special enough so that it is possible to compute them, but general enough so that they can stand for a universal collision in the above sense. A natural class of collisions to be considered is the class of generic successive collisions. Can we compute them ? Is there a universal collision among them ? A generic successive collision of three fat points is universal [3], ie. this collision has the same Hilbert function as the generic union of the three fat points. We use our theorem to compute the generic successive collision of four fat points. Our computation proves that this collision is not universal. Beyond this example, the computation also illustrates how our theorem can be used to determine many collisions, thus extending the toolbox of the Horace method.

# 2 Statement of the theorem

We fix a generically smooth quasi-projective scheme $X$ of dimension $d$, a locally free sheaf $L$ of rank one on $X$ and a sub-vector space $\mathcal{L} \subset H^0(X, L)$. Let $Z \subset X_{k(Z)}$ be a 0-dimensional subscheme parametrised by a non closed point of $Hilb(X)$ with residual field $k(Z)$. Let $\mathcal{L}(-Z) \subset \mathcal{L}$ be the sub-vector space of sections which vanish on $Z$ (see the definition below). Our goal is to give an estimate of the dimension $\dim \mathcal{L}(-Z)$ under suitable conditions.

A staircase $E \subset \mathbb{N}^d$ is a subset whose complement $C = \mathbb{N}^d \setminus E$ verifies $\mathbb{N}^d + C \subset C$. We denote by $I^E$ the ideal of $k[x_1, \dots, x_d]$ (resp. of $k[[x_1, \dots, x_d]]$), of $k[[x_1, \dots, x_d]][t] \dots$) generated by the monomials $x_1^{e_1}.\dots.x_d^{e_d} = x^e$ whose exponent $e = (e_1, \dots, e_d)$ is in $C$. If $E$ is a finite staircase, the subscheme $Z(E)$ defined by $I^E$ is 0-dimensional and its degree is $\#E$. The map $E \mapsto Z(E)$ is a one-to-one correspondance between the finite staircases of $\mathbb{N}^d$ and the monomial punctual subschemes of $Spec\ k[x_1, \dots, x_d]$. If $E = (E_1, \dots, E_s)$ is a set of finite staircases, if $X$ is irreducible and if $Z(E)$ is the (abstract non embedded) disjoint union $Z(E_1) \coprod \dots \coprod Z(E_s)$, there is an irreducible scheme

$P(E)$ which parametrizes the embeddings $Z(E) \to X_s$, where $X_s \subset X$ is the smooth locus ([6] and [7]). Such an embedding $Z(E) \to X_s$ determines a subscheme of $X$, thus there is a natural morphism $f : P(E) \to Hilb(X)$ to the Hilbert scheme of $X$. We denote by $X(E)$ the subscheme parametrised by $f(p)$ where $p$ is the generic point of $P(E)$. We will say that $X(E)$ is the generic union of the schemes $Z(E_1), \dots, Z(E_n)$. If $Z \subset X$ is a subscheme, denote by $\mathcal{L}(-Z) \subset \mathcal{L}$ the subvector space which contains the elements of $\mathcal{L}$ vanishing on $Z$. If $p$ is a non closed point of $Hilb(X)$ whose residual field is $k(p)$, and if $Z \subset X \times_k Spec\ k(p)$ is the corresponding subscheme, the definition of $\mathcal{L}(-Z)$ is as follows. Since $\mathcal{L} \otimes k(p) \subset H^0(L \otimes k(p), X \times k(p))$, it makes sense to consider the vector space $V \subset \mathcal{L} \otimes k(p)$ containing the sections which vanish on $Z$. Denoting by $\lambda$ the codimension of $V$, we may associate with $V$ a $k(p)$-point $g \in Grass_{k(p)}(\lambda, \mathcal{L} \otimes k(p)) = Grass_k(\lambda, \mathcal{L}) \times Spec\ k(p)$ ([5], prop.9.7.6). In particular $\mathcal{L}(-Z)$ is well defined as a (non closed) point of $Grass_k(\lambda, \mathcal{L})$. The goal of the theorem is to give an estimate of $\dim \mathcal{L}(-X(E))$.
To formulate the theorem, we need some combinatorial notations that we introduce now. The $k^{th}$ slice of a staircase $E \subset \mathbb{N}^d$ is the staircase $T(E,k) \subset \mathbb{N}^d$ defined by:

$$T(E,k) = \{(0, a_2, \dots, a_d) \textit{ such that } (k, a_2, \dots, a_d) \in E\}$$

If $E = (E_1, \dots, E_s)$ is a s-tuple of staircases and $t = (t_1, \dots, t_s)$, we set

$$T(E,t) = (T(E_1,t_1), T(E_2,t_2), \dots, T(E_s,t_s)).$$

A staircase $E \subset \mathbb{N}^d$ is characterized by a height function $h_E : \mathbb{N}^{d-1} \to \mathbb{N}$ which verifies:

$$\forall a, b \in \mathbb{N}^{d-1},\ h_E(a+b) \leq h_E(a)$$

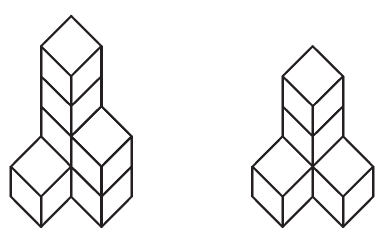
The staircase $E$ and $h_E$ can be deduced one from the other via the relation:

$$(a_1, \dots, a_d) \in E \Leftrightarrow a_1 < h_E(a_2, \dots, a_n)$$

The staircase $S(E,t)$ is defined by its height function:

$$\begin{aligned} h_{S(E,t)}(a_2, \dots, a_d) &= h_E(a_2, \dots, a_d) \text{ if } t \geq h_E(a_2, \dots, a_d) \\ &= h_E(a_2, \dots, a_d) - 1 \text{ if } t < h_E(a_2, \dots, a_d) \end{aligned}$$

Intuitivly, it is the staircase obtained by the suppression of the $t^{th}$ slice, as shown by the following figure.



Staircase      Suppression of slice number one

If $E=(E_1,\dots,E_s)$ is a family of staircases, and $t=(t_1,\dots,t_s)\in\mathbb{N}^s$, we put:

$$S(E,t)=(S(E_1,t_1),S(E_2,t_2),\dots,S(E_s,t_s)).$$

If $(t_1,\dots,t_r)\in(\mathbb{N}^s)^r$, the recursive formula

$$S(E,t_1,\dots,t_r)=S(S(E,t_1,\dots,t_{r-1}),t_r)$$

defines the s-tuple of staircases $S(E,t_1,\dots,t_r)$ obtained from the s-tuple $E=(E_1,\dots,E_s)$ by suppression of $r$ slices in each $E_i$.
If $p\in X$ is a smooth point, a formal neighborhood of $p$ is a morphism $\varphi: \mathit{Spec}\ k[[x_1,\dots,x_d]]\to X$ which induces an isomorphism between $\mathit{Spec}\ k[[x_1,\dots,x_d]]$ and the completion $\widehat{\mathcal{O}_p}$ of the local ring of $X$ at $p$. If $p=(p_1,\dots,p_s)$ is a s-tuple of smooth distinct points, a formal neighborhood of $p$ is a morphism $(\varphi_1,\dots,\varphi_s):U\to X$ from the disjoint union $U=V_1\coprod\dots\coprod V_s$ of $s$ copies of $\mathit{Spec}\ k[[x_1,\dots,x_d]]$ to $X$, where $\varphi_i:V_i\to X$ is a formal neighborhood of $p_i$. If $D$ is a divisor on $X$, we say that $\varphi$ and $D$ are compatible if $D$ is defined by the equation $x_1=0$ around each $p_i$ (in particular, $p_i$ is a smooth point of $D$).
Consider the translation morphism:

$$\begin{array}{rcl} Tr_{v_1}:k[[x_1,\dots,x_d]] & \to & k[[x_1,\dots,x_d]]\otimes k[[t]] \\ x_1 & \mapsto & x_1\otimes 1-1\otimes t^{v_1} \\ x_i & \mapsto & x_i\otimes 1 \text{ si } i>1 \end{array}$$

If $E_1$ is a staircase, the ideal

$$J(E_1,v_1)=Tr_{v_1}(I^{E_1})k[[x_1,\dots,x_d]]\otimes k[[t]]\subset k[[x_1,\dots,x_d]]\otimes k[[t]]$$

defines a flat family $F_1$ of subschemes of $\mathit{Spec}\ k[[x_1,\dots,x_d]]$ parametrised by $\mathit{Spec}\ k[[t]]$. This corresponds geometrically to the family whose fiber over $t$ is obtained from $V(I^{E_1})$ by the translation $x_1\mapsto x_1-t^{v_1}$. If $\varphi_1$ is a formal neighborhood of $p_1$, $F_1$ can be seen as a flat family of subschemes of $X$ via $\varphi_1$, thus it defines a morphism $\mathit{Spec}\ k[[t]]\to \mathit{Hilb}(X)$. We denote by $X(\varphi_1,E_1,t,v_1)$ the non closed point of $\mathit{Hilb}(X)$ parametrised by the image of the generic point. The first coordinate does not play any specific role, thus more generally, if $E=(E_1,\dots,E_s)$ is a family of staircases, if $\varphi=(\varphi_1,\dots,\varphi_s)$ is a formal neighborhood of $(p_1,\dots,p_s)$, if $v=(v_1,\dots,v_s)\in\mathbb{N}^s$, one defines similarly families $F_i\subset X\times \mathit{Spec}\ k[[t]]$ flat over $\mathit{Spec}\ k[[t]]$. Since $F_i\cap F_j=\emptyset$ for $i\neq j$, the union $F=F_1\cup\dots\cup F_s$ is still flat over $\mathit{Spec}\ k[[t]]$ and corresponds to a morphism $\mathit{Spec}\ k[[t]]\to \mathit{Hilb}(X)$. We denote by $X_\varphi(E,t,v)$ the image of the generic point and by $X_\varphi(E)=X_\varphi(E,0,v)$ the image of the special point (which does not depend on $v$). Finally, we denote by $[x]$ the integer part of a real $x$.
We are now ready to state the theorem. By the above, $\mathcal{L}(-X_\varphi(E,t,v))$ corresponds to a morphism $\mathit{Spec}\ k((t))\to\mathbb{G}$ to a Grassmannian $\mathbb{G}$, which can be extended to a morphism $\mathit{Spec}\ k[[t]]\to\mathbb{G}$ by valuative properness. The theorem gives a control of the limit obtained under suitable conditions.

**Theorem 1.** *Let $D$ be an effective divisor on a quasi-projective scheme $X$, $p = (p_1, \dots, p_s)$ be a $s$-tuple of smooth points of $X$, $\varphi$ a formal neighborhood of $p$ compatible with $D$, $v = (v_1, \dots, v_s) \in \mathbb{N}^s$ a speed vector, $E = (E_1, \dots, E_s)$ be staircases and $X_\varphi(E, t, v)$ the generic union of subschemes defined by $\varphi$. Suppose that one can find integers $n_1 > \dots > n_r$ such that:*

- *$\forall k$, $n_k - n_{k+1} \geq max(v_i)$,*
- *$\forall i$, $1 \leq i \leq r$, $\mathcal{L}(-(i-1)D - Z_i) = \mathcal{L}(-iD)$*

*where $t_i = ([\frac{n_i}{v_1}], \dots, [\frac{n_i}{v_s}])$, $T_i = T(E, t_i)$ and $Z_i = X_\varphi(T_i)$. Then*

$$\lim_{t \to 0} \mathcal{L}(-X_\varphi(E, t, v)) \subset \mathcal{L}(-rD - X_\varphi(S(E, t_1, \dots, t_r)))$$

**Remark 2.** *The main lemma 2.3 of [1] corresponds essentially to the above theorem with $r = 1$.*

If $X$ is irreducible, $X(E)$ is well defined and it specializes to $X_\varphi(E, t, v)$. Thus we get by semi-continuity the inequality

$$\dim \mathcal{L}(-X(E)) \leq \dim \mathcal{L}(-X_\varphi(E, t, v)) = \dim \lim_{t \to 0} \mathcal{L}(-X_\varphi(E, t, v)).$$

Combining this inequality with the theorem, we obtain the following estimate of $\dim \mathcal{L}(-X(E))$ in terms of a linear system of smaller degree.

**Corollary 3.** $\dim \mathcal{L}(-X(E)) \leq \dim \mathcal{L}(-rD - X_\varphi(S(E, t_1, \dots, t_r)))$

**Remark 4.** *In case $\mathcal{L}$ is infinite dimensional, the theorem still makes sense since Grassmannians of finite codimensional vector spaces of $\mathcal{L}$ are still well defined and the limit makes sense in such a Grassmannian.*

# 3 Proof of theorem 1

We start with an informal explanation of the ideas in the proof in the case $s = 1$. Suppose that we have of family of sections $s(t)$ of $L$ which vanish on a moving punctual subscheme $Z(t) = X_\varphi(E, t, v)$ whose support $p(t)$ tends to $p(0)$ as $t$ tends to 0. Using local coordinates around $p(0)$, the sections of $L$ can be considered as functions and the vanishing on $Z(t)$ translates to $s(t) \in J(t)$ where $J(t)$ is the ideal of $Z(t)$. Denote by $J_{n_1}$ the restriction of $J(t)$ to the infinitesimal neighborhood *Spec* $k[t]/t^{n_1}$ of $t = 0$. Suppose that the family of sections over *Spec* $k[t]/t^{n_1}$ is a family of sections which vanish on $Z_1$. Then it is a family of sections vanishing on $D$ since by hypothesis a section which vanish on $Z_1$ automatically vanishes on $D$. If $D$ is defined locally by the equation $x_1 = 0$, this means that $s(t) = x_1 s'(t)$ with $s'(t) \in (J_{n_1} : x_1)$. Restrict now to the smaller infinetismal neighborhood *Spec* $k[t]/t^{n_2}$. Suppose that over this restriction, the family of sections, which already vanish on $D$, vanish also on $Z_2$ (i.e. $s'(t)$ is a family of sections vanishing on $Z_2$). Then by hypothesis, the sections vanish twice on $D$. Using local coordinates, this means that $s(t) = x_1^2 s''(t)$ with $s''(t) \in ((J_{n_1} : x_1)_{n_2} : x_1)$. After several restrictions, we put $t = 0$ and we get

$s(0) = x_1^r s^{(r)}(0)$ where $s^{(r)}(0)$ is in a prescribed ideal. The control we get in this way of the element $s(0) \in \lim_{t\to 0} \mathcal{L}(-X_\varphi(E,t,v))$ translates into the inclusion

$$\lim_{t\to 0} \mathcal{L}(-X_\varphi(E,t,v)) \subset \mathcal{L}(-rD - X_\varphi(S(E,t_1,\dots,t_r)))$$

given by the theorem.
To play the above game, one needs to be able to compute in the successive steps ideals like $((J_{n_1} : x_1)_{n_2} : x_1)$ defined using restrictions and transporters. In view of this explanation, one can understand the conditions on the $n_i$ of the theorem as follows. The condition $n_1 \geq n_2 \geq n_3 \dots$ comes from the fact that we restrict successivly to smaller and smaller neighborhoods. The condition $n_k - n_{k+1} \geq max(v_i)$ is a technical condition to be able to compute the successive ideals defined via transporters and restrictions.

Let us start the proof itself now. In the context of the theorem, we are given a set of staircases $E = (E_1, \dots, E_s)$, a vector $v = (v_1, \dots, v_s)$, a divisor $D$ and a formal neighborhood $\varphi$ of $(p_1, \dots, p_s)$ in which $D$ is given by the equation $x_1 = 0$ around each $p_i$. For $n > 0$, we put $R_n = k[[x_1, \dots, x_d]]^s \otimes k[[t]]/(t^n)$ and $R_\infty = k[[x_1, \dots, x_d]]^s \otimes k[[t]]$. We denote by $\psi_{np} : R_n \to R_p$ the natural projections, which exist for $p \leq n \leq \infty$. If $J \subset R_\infty$ is an ideal, we define we define recursively the ideals $J_{n_1:n_2:\dots:n_k} \subset R_{n_k}$ and $J_{n_1:n_2:\dots:n_k:} \subset R_{n_k}$ using transporters and restrictions by the formulas

- $J_{n_1} = \psi_{\infty n_1}(J)$,
- $J_{n_1:n_2:\dots:n_k:} = (J_{n_1:n_2:\dots:n_k} : x_1)$,
- $J_{n_1:n_2:\dots:n_k} = \psi_{n_{k-1}n_k}(J_{n_1:n_2:\dots:n_{k-1}:})$

As explained above, the vector space $\mathcal{L}(-X(\varphi,E,t,v))$ corresponds to a morphism $Spec\ k((t)) \to \mathbb{G}$ (where $\mathbb{G}$ is a Grassmannian of subvectors spaces of $\mathcal{L}$) which extends to a morphism $Spec\ k[[t]] \to \mathbb{G}$. The universal family over the Grassmannian $\mathbb{G}$ pulls back to a family $U \subset Spec\ k[[t]] \times \mathcal{L}$. Let $e_i$ be a local generator of $L$ at $p_i$. Any section $\sigma$ of the line bundle $L$ can be written down $\sigma = \sigma_i e_i$ around $p_i$ for some $\sigma_i \in k[[x_1, \dots, x_d]]$. The map:

$$\begin{array}{ccc} \mathcal{L} & \to & k[[x_1,\dots,x_d]]^s \\ \sigma & \mapsto & (\sigma_1,\dots,\sigma_s) \end{array}$$

identifies $U$ with a subscheme of $Spec\ k[[t]] \times k[[x_1, \dots, x_d]]^s$. The theorem will be proved if we show that the special fiber $U(0)$ contains only sections vanishing $r$ times on $D$ and if, in local coordinates, $U(0)$ is included in $x_1^r I^{S(E,t_1,\dots,t_r)}$.
Let us denote by $U_{n_i}$ the restriction of $U$ over the subscheme $Spec\ k[[t]]/t^{n_i}$. We show by induction that:

$$\forall i \geq 1,\ U_{n_i} \subset x_1^i J_{n_1:n_2:\dots:n_i:}$$

where $J = J(E_1, v_1) \oplus \dots \oplus J(E_s, v_s) \subset R_\infty$. The fibers of $U$ contain sections of $\mathcal{L}$ which vanish on $X_\varphi(E,t,v)$. Since $J$ is the ideal of $X_\varphi(E,t,v)$, this implies the inclusion $U \subset J$, hence $U_{n_1} \subset J_1$. By corollary 8, this inclusion implies that the fibers of $U_{n_1}$ are elements of $\mathcal{L}$ which vanish on $Z_1$, hence they vanish on

$D$ by hypothesis. It follows that elements of $U_{n_1}$ are dividible by $x_1$ and we can then write: $U_{n_1} \subset x_1 J_{n_1:}$ . Suppose now that $U_{n_i} \subset x_1^i J_{n_1:n_2:\dots:n_i:}$ . Then $U_{n_{i+1}} \subset x_1^i J_{n_1:n_2:\dots:n_i:n_{i+1}}$. By corollary 8, this inclusion implies that the fibers of $U_{n_{i+1}}$ are elements of $\mathcal{L}(-iD)$ which vanish on $Z_{i+1}$, hence they vanish on $D$ by hypothesis. It follows that elements of $U_{n_{i+1}}$ are dividible by $x_1^{i+1}$ and we can write $U_{n_{i+1}} \subset x_1^{i+1} J_{n_1:n_2:\dots:n_{i+1}:}$ . This ends the induction on $i$. In particular, for $i = r$, using corollary 9 for the last equality, we have the required inclusion:

$$U(0) = U_{n_r}(0) \subset x_1^r J_{n_1:n_2:\dots:n_r:}(0) = x_1^r I^{S(E,t_1,\dots,t_r)}. \blacksquare$$

We now turn to the proof of the corollaries 8 and 9 on which the above proof relies. Note that $J = (J^1, \dots, J^s)$ and $I^{T_k} = ((I^{T_k})^1, \dots, (I^{T_k})^s)$ are defined componentwise, the component number $i$ corresponding to the study around the point $p_i$. Thus corollary 8 and 9 below can be proved for each component and one may suppose $s = 1$ to prove it. We thus suppose for the rest of this section that $s = 1$, that $E = (E_1, \dots, E_s)$ is a staircase given by a height function $h$, and that $v = (v_1, \dots, v_s) \in \mathbb{N}$.
Let $\mathcal{B}$ (resp. $\mathcal{C}$) be the set of elements $m = (m_2, \dots, m_d) \in \mathbb{N}^{d-1}$ such that $h(m) \neq 0$ (resp. $h(m) = 0$). Remark that $\mathcal{B}$ is finite due to the finitness of $E$. We denote by

- $C(t) \subset R_n$ the $k[[x_1]] \otimes k[[t]]$ sub-module containing the elements $\sum a_{m_1 m_2 \dots m_d} x_1^{m_1} x_2^{m_2} \dots x_d^{m_d} \otimes f(t)$, where $f(t) \in k[[t]]/t^n$ and $(m_2, \dots, m_d) \in \mathcal{C}$
- $C(0) \subset R_1 = k[[x_1, \dots, x_d]]$ the $k[[x_1]]$ sub-module containing the series $\sum a_{m_1 m_2 \dots m_d} x_1^{m_1} x_2^{m_2} \dots x_d^{m_d}$ where $(m_2, \dots, m_d) \in \mathcal{C}$
- $B(m) \subset R_n$ the $k[[x_1]] \otimes k[[t]]$ sub-module generated by $f_m = (x_1 - t^v)^{h(m)} x_2^{m_2} \dots, x_d^{m_d}$,
- $B(m, 0) \subset R_1 = k[[x_1, \dots, x_d]]$ the $k[[x_1]]$ sub-module generated by $f_m(0) = (x_1)^{h(m)} x_2^{m_2} \dots, x_d^{m_d}$,
- $B_{n_1 n_2 \dots n_k}(m) \subset R_n$ the $k[[x_1]] \otimes k[[t]]$ sub-module generated by the elements $f_m$, $\frac{t^{\alpha_{k-i+1}} f_m}{x_1^i}$, $1 \leq i \leq k$, where $\alpha_i = max(0, n_i - vh(m))$ for $i > 0$. In particular, for $k = 0$, $B_{n_1 n_2 \dots n_k}(m) = B(m)$.

To simplify the notations, we have adopted above the same notation for distinct submodules (leaving in distinct ambiant modules). The following lemma says that the module $B_{n_1 n_2 \dots n_k}(m)$ is well defined as a sub-module of $R_j$ for $j \leq n_k$.

**Lemma 5.** *Let $j \leq n_k$. If $i \leq k$, the element $\frac{t^{\alpha_{k-i+1}} f_m}{x_1^i} \in R_j$. In particular $B_{n_1 n_2 \dots n_k}(m) \subset R_j$ is well defined for $j \leq n_k$. If in addition, $j \leq n_{k+1}$, then $\frac{t^{\alpha_{k-i+1}} f_m}{x_1^i}$ is a multiple of $x_1$.*

*Proof.* First, if $l < i$, the coefficient of $x_1^l$ in $t^{\alpha_{k-i+1}} f_m$ is a multiple of $t^{\alpha_{k-i+1}} t^{v(h(m)-l)}$. This term is zero in $R_j$ since the exponent of $t$ is at least $n_{k-i+1} - vl \geq n_k + (i-1)v - vl \geq n_k \geq j$. It follows that $\frac{t^{\alpha_{k-i+1}} f_m}{x_1^i} \in R_j$ is well defined. A similar estimate shows that for $l \leq i$, the coefficient of

$x_1^l$ in $t^{\alpha_{k-i+1}} f_m$ is zero in $R_j$ for $j \leq n_{k+1}$. Thus $\frac{t^{\alpha_{k-i+1}} f_m}{x_1^i}$ is a multiple of $x_1$. ■

**Lemma 6.**
- *As $k[[x_1]]$-modules, $I^E = \bigoplus_{m\in\mathcal{B}} B(m,0) \oplus C(0) \subset k[[x_1,\dots,x_d]]$*
- *As $k[[x_1]] \otimes k[[t]]$-modules, $J = \bigoplus_{m\in\mathcal{B}} B(m) \oplus C(t) \subset R_n$*

*Proof*: This is a straightforward verification left to the reader. ■

**Lemma 7.** *We have the equality of $k[[x_1]] \otimes k[[t]]$-modules:*
- $J_{n_1:\cdots:n_k} = \bigoplus_{m\in\mathcal{B}} B_{n_1 n_2\dots n_{k-1}}(m) \oplus C(t) \subset R_{n_k}$
- $J_{n_1:\cdots:n_k:} = \bigoplus_{m\in\mathcal{B}} B_{n_1 n_2\dots n_k}(m) \oplus C(t) \subset R_{n_k}$

*Proof.* Let us say that the number of indexes of $J_{n_1:\cdots:n_k}$ and $J_{n_1:\cdots:n_k:}$ is respectivly $2k-1$ and $2k$. We prove the lemma by induction on the number $i$ of indexes. If $i=1$, we get from the preceding lemma the equality

$$\begin{aligned} J_{n_1} = \psi_{\infty n_1}(J) &= \sum_{m\in\mathcal{B}} \psi_{\infty n_1}(B(m)) + \psi_{\infty n_1}(C(t)) \\ &= \sum_{m\in\mathcal{B}} B(m) + C(t) \text{ in } R_{n_1}. \end{aligned}$$

The last sum is obviously direct, thus it is the required equality.
Suppose now that we want to prove the lemma for $i = 2k-1$. This is exactly the same reasoning as in the case $i=1$, substituting $J_{n_1:\cdots:n_k}$, $J_{n_1:\cdots:n_{k-1}:}$ and $\psi_{n_{k-1}n_k}$ for $J_{n_1}$, $J$, and $\psi_{\infty,n_1}$.
For the last case $i=2k$. Taking the transporter from the expression of $J_{n_1:\cdots:n_k}$ coming from induction hypothesis, we get:

$$J_{n_1:\cdots:n_k:} = \bigoplus_{m\in\mathcal{B}} (B_{n_1 n_2\dots n_{k-1}}(m) : x_1) \oplus (C(t) : x_1)$$

The equality $(C(t) : x_1) = C(t)$ is obvious, so we are done if we prove the equality $(B_{n_1 n_2\dots n_{k-1}}(m) : x_1) = B_{n_1 n_2\dots n_k}(m)$ in the ambiant module $R_{n_k}$. The inclusion $\supset$ is clear since for every generator $g$ of $B_{n_1 n_2\dots n_k}(m)$, $x_1 g$ is a multiple of one of the generators of $B_{n_1 n_2\dots n_{k-1}}(m)$. As for the reverse inclusion, if $z \in (B_{n_1 n_2\dots n_{k-1}}(m) : x_1)$, one can write down

$$x_1 z = \sum_{1\leq i\leq k-1} P_i \frac{t^{\alpha_{k-i+1}} f_m}{x_1^i} + x_1 P_0 f_m + Q_0 f_m \quad (*)$$

where $P_i \in k[[x_1]] \otimes k[[t]]$ and $Q_0 \in k[[t]]$. By lemma 5, the terms $\frac{t^{\alpha_{k-i+1}} f_m}{x_1^i} \in R_{n_k}$ are dividible by $x_1$, thus $x_1$ divides $Q_0 f_m$. It follows that the coefficient $Q_0 t^{vh(m)} x_2^{m_2} \dots x_d^{m_d}$ of $x_1^0$ in $Q_0 f_m$ is zero, which happens only if

$Q_0$ is a multiple of $t^{max(0,n_k-vh(m))} = t^{\alpha_k}$. Writing down $Q_0 = t^{\alpha_{k-1}+1}$ and dividing the displayed equality $(*)$ by $x_1$ shows that $z \in B_{n_1n_2\dots n_k}$, as expected. ■

**Corollary 8.** $J_{n_1:n_2:\dots:n_k} \subset I^{T_k}$

*Proof.* In view of the previous lemma, and since the inclusion $C \subset I^{T_k}$ is obvious, one simply has to check that the generators of $B_{n_1:n_2:\dots:n_k}(m)$ verify the inclusion. The generators are explicitly given thus this is a straightforward verification. ■

**Corollary 9.** $J_{n_1:n_2:\dots:n_k}(0) = I^{S(E,t_1,\dots,t_k)}$.

*Proof.* According to lemmas 7 and 6, it suffices to show that $B_{n_1n_2\dots n_k}(m,0) \subset k[[x_1]]$ is the submodule generated by $x_1^{h(m)-p(m)}$ where $p(m)$ is the number of $t_i$'s verifying $t_i < h(m)$. Since the generators of $B_{n_1n_2\dots n_k}(m)$ are explicitely given, the corollary just comes from the evaluation of these generators at $t = 0$. ■

# 4 The Hilbert function of $k^2$ fat points in $\mathbb{P}^2$

In this section, we compute the Hilbert function of the generic union of $k^2$ fat points in $\mathbb{P}^2$ of the same multiplicity $m$.
We work over a field of characteristic 0.

**Definition 10.** *If $Z \subset \mathbb{P}^2$ is a zero-dimensional subscheme of degree $deg(Z)$, we denote by $H_v(Z) : \mathbb{N} \to \mathbb{N}$ the virtual Hilbert function of $Z$ defined by the formula $H_v(Z,d) = min(\frac{(d+1)(d+2)}{2}, deg(Z))$. The critical degree for $Z$, denoted by $d_c(Z)$ is the smallest integer $d$ such that $H_v(Z,d) > deg(Z)$.*

**Theorem 11.** *Let $Z$ be the generic union of $k^2$ fat points of multiplicity $m$. Then $H(Z) = H_v(Z)$.*

Let us recall the following well known lemma:

**Lemma 12.** *If $H(Z,d) \geq H_v(Z,d)$ for $d = d_c(Z)$ and $d = d_c(Z) - 1$, then $H(Z) = H_v(Z)$.*

**Definition 13.** *The regular staircase $R_m \subset \mathbb{N}^2$ is the set defined by the relation $(x,y) \in R_m \Leftrightarrow x + y < m$. A quasi-regular staircase $E$ is a staircase such that $R_m \subset E \subset R_{m+1}$ for some $m$. A right specialized staircase is a staircase such that $((x,y) \in E$ and $y > 0) \Rightarrow (x+1, y-1) \in E$. A monomial subscheme of $\mathbb{P}^2$ with staircase $E$ is a punctual subscheme supported by a point $p$ which is defined by the ideal $I^E$ in some formal neighborhood of $p$.*

Our first intermediate goal is lemma 15 which says that under suitable conditions, if $Z = L \cup R \subset \mathbb{P}^2$ is a subscheme with $L$ included in a line, the Hilbert function of $Z$ is determined by that of $R$.

**Proposition 14.** *Let $Z$ be a generic union of fat points. The following conditions are equivalent.*

- $H(Z) = H_v(Z)$
- *there exists a quasi-regular right-specialized staircase $E$ and a collision $C$ of the fat points which is monomial with staircase $E$.*
- *there exists a quasi-regular staircase $E$ and a collision $C$ of the fat points which is monomial with staircase $E$.*

*Proof.* $1 \Rightarrow 2$. Let $\rho_t$ be the automorphism of $\mathbb{P}^2 = Proj(k[X,Y,H])$ defined for $t \neq 0$ by $f_t : X \mapsto \frac{X}{t}, Y \mapsto \frac{Y}{t}, H \mapsto H$. Consider the collision $C = \lim_{t\to 0} f_t(Z)$. It is a subscheme of the affine plane $Spec\ k[x = \frac{X}{H}, y = \frac{Y}{H}]$ supported by the origin $(0,0)$. It is shown in [4] that if $H(Z) = H_v(Z)$, then there is an integer $m$ such that the ideal of $C$ verifies $I^{R_{m+1}} \subset I(C) \subset I^{R_m}$. Thus $I(C) = V \oplus k[x,y]_{\geq m+1}$ where $k[x,y]_{\geq m+1}$ stands for the vector space generated by the monomials of degree at least $m+1$, and $V \subset k[x,y]_m$. Let now $g_t : x \mapsto x - ty, y \mapsto y$. Then the ideal of $D = \lim_{t\to\infty} g_t(C)$ is $I(D) = W \oplus k[x,y]_{\geq m+1}$ where $W = \lim_{t\to\infty} g_t(V)$ is a vector space which admits a base of the form $y^m, xy^{m-1}, \ldots, x^k y^{m-k}$. Thus $I(D) = I^E$ for some quasi-regular right-specialized staircase $E$. And $D$ is a collision of the fat points since it is a specialisation of the collision $C$ and since being a collision is a closed condition.
$2 \Rightarrow 3$ is obvious.
$3 \Rightarrow 1$. If there exists a collision $C$ associated with a quasi-regular staircase $E$, then by semi-continuity $H(Z,d) \geq H(C,d) = min(\frac{(d+1)(d+2)}{2}, \#E) = min(\frac{(d+1)(d+2)}{2}, deg(C)) = min(\frac{(d+1)(d+2)}{2}, deg(Z)) = H_v(Z,d)$. Since the well known reverse inequality $H_v(Z,d) \geq H(Z,d)$ is always true, we have the required equality $H_v(Z,d) = H(Z,d)$. ■

**Lemma 15.** *Let $R \subset \mathbb{P}^2$ be a generic union of fat points, $D \subset plp$ be a generic line, $L \subset D$ be a subscheme whose support is generic in $D$. Let $Z = R \cup L$ and suppose that the degree of $L$ satisfies $deg(L) \leq d_c(R)$. Then $H(R) = H_v(R)$ implies $H(Z) = H_v(Z)$.*

*Proof.* By the above lemma and its proof, there exists a quasi-regular right specialized staircase $E$ and a collision $C$ of the fat points supported by the origin of $\mathbb{A}^2 = Spec\ k[x,y]$ such that the ideal of $C \subset \mathbb{A}^2$ is $I(C) = I^E$. By the genericity hypothesis, $L$ can be specialized to the subscheme $L(t)$ with equation $(y - t, x^{deg(L)})$. Obviously $L(t)$ is monomial with staircase $F = \{(0,0), (1,0), \ldots, (r,0)\}$. Let $D = \lim_{t\to 0} C \cup L(t)$. By [7], $I(D) = I^G$ for some monomial staircase $G$. Moreover, the explicit description of $G$ given in [7] ( $G$ is the "vertical collision" of $E$ and $F$) shows that $G$ is quasi-regular.

Since $Z = R \cup L$ can be specialized to a scheme $D$ defined by a quasi regular staircase, $H(Z) = H_v(Z)$. ■

**Lemma 16.** *Let $Z \subset \mathbb{P}^2$ be a union of $k^2$ fat points of multiplicity $m$ with $k \geq 4$. The critical degree $d_c(Z)$ verifies $km + 1 < d_c(Z) \leq km + k - 2$.*

*Proof*: Direct calculation. ■

*Proof of theorem 11.*
We show by induction on $k$ that the Hilbert function of the generic union $Z$ of $k^2$ fat points of multiplicity $m$ is the virtual Hilbert function $H_v(Z)$. If $k \leq 3$, this is known by [9]. So we may suppose $k \geq 4$. According to lemma 12, we only need to check that $H(Z,d) \geq H_v(Z,d)$ for $d = d_c(Z)$ or $d = d_c(Z)-1$, and, by lemma 16, such a $d$ verifies $d = km + s$ for some $s$ satisfying $0 \leq s \leq k-2$. By semi-continuity, it suffices to specialize $Z$ to a scheme $Z'$ with $H(Z',d) \geq H_v(Z,d)$. First, we choose a generic line $D$ and generic points $p_1, \ldots, p_{2k-1}$ on $D$. We divide the $k^2$ fat points into three subsets $E_1, E_2, E_3$ of respective cardinal $k, k-1, (k-1)^2$. We specialize the $k$ fat points of $E_1$ on the points $p_k, \ldots, p_{2k-1}$. We leave the generic $(k-1)^2 + (k-1)$ points of $E_3 \cup E_2$ in their generic position. We denote by $\mathcal{L}$ the set of sections of $\mathcal{O}(d)$ which vanish on the fat points of $E_1 \cup E_3$. Since the points of $E_1$ have been specialised, we have by semi-continuity the inequality:

$$(*)\ H(Z,d) \geq \frac{(d+1)(d+2)}{2} - \dim \mathcal{L}(-X(E))$$

where

$$E = (\underbrace{R_m, \ldots, R_m}_{(k-1)\ copies}).$$

We now make a further specialisation, moving the $k-1$ fat points of $E_2$ on the points $p_1, \ldots, p_{k-1}$ using theorem 1. To this end, we fix the notations. We choose a formal neighborhood $\varphi$ of $p = (p_1, \ldots, p_{k-1})$, a number $N >> 0$ and we take the speed vector

$$v = (\underbrace{N, \ldots, N}_{k-s-2\ times}, \underbrace{N+1, \ldots, N+1}_{s+1\ times}).$$

Finally, we let

$$n_i = (N+1)(m-i+1) - 1, 1 \leq i \leq m.$$

Let us check that the conditions of theorem 1 apply. The condition $n_k - n_{k+1} \geq max(v_i)$ is obviously satisfied. As for the remaining condition, remark that $\mathcal{L}(-(i-1)D)$ is a set of sections of $\mathcal{O}(d-i+1)$ which vanish on $p_k^{m-i+1}, \ldots, p_{2k-1}^{m-i+1}$. In particular, if $Z_i$ is a punctual subscheme of $D$ of cardinal $d - i + 2 - k(m-i+1) = s + 1 + (i-1)(k-1)$ whose support does not meet

the union $p_k \cup \cdots \cup p_{2k-1}$, then $\mathcal{L}(-iD - Z_i) = \mathcal{L}(-(i+1)D)$. In our case, $Z_i$ is a union of one-dimensional fat points of the line $D$. Let us compute its degree. The subscheme $Z_i$ is supported by $p_1 \cup \cdots \cup p_{k-1}$ and we denote by $d_j$ the degree of the part $(Z_i)_{p_j}$ supported by $p_j$. It is the cardinal $m - [\frac{n_i}{v_j}]$ of the slice $T(R_m, [\frac{n_i}{v_j}])$, that is $d_j = i-1$ if $j \leq k-s-2$ and $d_j = i$ if $k-s-1 \leq j \leq k-1$. Thus the degree of $Z_i$ is the sum of the $d_j$, that is $s+1+(i-1)(k-1)$. We can then apply theorem 1 and its corollary. We conclude that:

$$(**)\ \dim \mathcal{L}(-X(E)) \leq \dim \mathcal{L}(-mD - X_\varphi(S(E, t_1, \ldots, t_m)).$$

The linear system $\mathcal{L}(-mD)$ is the set of sections of $\mathcal{O}(d-m)$ which vanish on the union $Z'$ of the fat points of $E_3$. Moreover, $X_\varphi(S(E, t_1, \ldots, t_m))$ is the union $L$ of the one-dimensional fat points of $p_1^m \cap D, \ldots, p_{k-s-2}^m \cap D$. It follows that

$$(***)\ \dim \mathcal{L}(-mD - X_\varphi(S(E, t_1, \ldots, t_m)) = \frac{(d-m)(d-m+1)}{2} H(Z' \cup L, d-m)$$

. By lemma 15 and the induction, we have

$$(****)\ H(Z' \cup L, d-m) = H_v(Z' \cup L, d-m)$$

Now, by construction (or by an easy direct calculation),

$$(*****)\ H_v(Z' \cup L, d-m) - \frac{(d-m)(d-m+1)}{2} = H_v(Z, d) - \frac{d(d+1)}{2}$$

Putting together the displayed equalities and inequalities (*)…(*****) gives the required inequality $H(Z, d) \geq H_v(Z, d)$. ■

# 5 Collisions of fat points

We start with a definition of a generic successive collision of fat points in $\mathbb{A}^2$. We proceed by induction. A generic successive collision of one fat point $p^m$ is the fat point itself. Suppose defined the generic successive collision $Z_{m_1 \ldots m_{k-1}}$ of $p_1^{m_1}, \ldots, p_{k-1}^{m_{k-1}}$. Let $C(d)$ be the generic curve of degree $d$ containing the support $O$ of $Z_{m_1 \ldots m_{k-1}}$. Let

$$Z_{m_1 \ldots m_k}(d) = \lim_{p \in C(d),\ p \to O} Z_{m_1 \ldots m_{k-1}} \cup p^{m_k}.$$

**Proposition 17.** *There exists an integer $d_0$ such that $\forall d \geq d_0$, $Z_{m_1 \ldots m_k}(d) = Z_{m_1 \ldots m_k}(d_0)$. We denote this subscheme by $Z_{m_1 \ldots m_k}$ and this is by definition the generic successive collision of $p_1^{m_1}, \ldots, p_k^{m_k}$.*

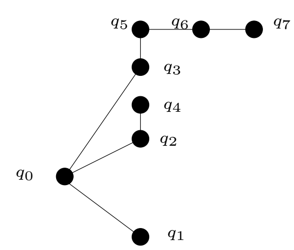
*Proof.* Consider the morphism $f : \mathbb{A}^2 \setminus \{O\} \to Hilb(\mathbb{A}^2) \subset Hilb(\mathbb{P}^2)$ which sends the point $p \in \mathbb{A}^2$ to the subscheme $Z_{m_1 \ldots m_{k-1}} \cup p^{m_k}$. It extends to a morphism $\tilde{f}$ :

$S \to Hilb(\mathbb{P}^2)$, where $\pi : S \to \mathbb{A}^2$ is a composition of blowups (of simple points). The embeddings $Spec\ k[t]/(t^d) \to \mathbb{A}^2$ sending the support of $Spec\ k[t]/(t^d)$ to $O \in \mathbb{A}^2$ form an irreducible variety and we denote by $g : Spec\ k[t]/t^d \to \mathbb{A}^2$ the corresponding generic embedding. For $p \geq d$, the intersection $C(p) \cap O^d$ of the curve with the fat point is isomorphic as an abstract scheme to $Spec\ k[t]/(t^d)$; since for any embedding $i : Spec\ k[t]/(t^d) \to \mathbb{A}^2$, there exists a curve of degree $d$ which contains the image $Im(i)$, it follows that $C(p) \cap O^d$ is the subscheme associated with the generic embedding $g$. In particular, $C(p) \cap O^{d_0} = C(d_0) \cap O^{d_0}$ if $p \geq d_0$. Choose $d_0 > n$ where $n$ is the number of blowups in $\pi$. Since the order of contact of $C(p)$ and $C(d_0)$ is at least $d_0$, the number of blowups is not sufficient to separate the curves and the strict transforms $\tilde{C}(p) \subset S$ and $\tilde{C}(d_0) \subset S$ intersect in a point $s$. It follows that $Z_{m_1 \dots m_k}(p) = Z_{m_1 \dots m_k}(d_0) = \tilde{f}(s)$. ■

Our goal is to compute the generic collision $Z_{mmmm}$ of 4 fat points of multiplicity $m$.

**Remark 18.** *With the notations of proposition 17, the integers $d_0$ which appear in the definition of $Z_{mmmm}$ will always be equal to 1. In other words, the collision will be shown to depend only on the tangent directions of the approaching fat points.*

We will describe $Z_{mmmm}$ as a pushforward via a blowup $\pi : \tilde{S} \to \mathbb{A}^2$, where $\pi$ is the blowup defined by the following Enriques diagram .




We recall for convenience what this means. Let $q_0 \in \mathbb{A}^2$, $q_1, q_2, q_3$ be three distinct tangent directions at $q_0$. Let

$$\eta : S_1 \to S_0 = \mathbb{A}^2$$

be the blowup of $q_0$, and $Q_0 \subset S_1$ the exceptional divisor. Let

$$S_2 \to S_1$$

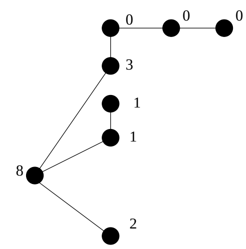
be the blowup of $(q_1 \cup q_2 \cup q_3) \subset Q_0$, and $Q_1, Q_2, Q_3 \subset S_2$ the respective exceptional divisors. If $Q_i \subset S_{n_i}$ is an exceptional divisor, and if $S_j \to S_{n_i}$ is a sequence of blowups, we still denote by $Q_i \subset S_j$ (resp. we denote by $E_i \subset S_j$) the strict transform (resp. the total transform) of $Q_i$ in $S_j$. With this convention, let $q_4 = Q_0 \cap Q_2 \in S_2$, $q_5 = Q_0 \cap Q_3 \in S_2$. Let

$$S_3 \to S_2$$

be the blowup of $q_4 \cup q_5$, $Q_4, Q_5$ the corresponding exceptionnal divisors. Let $q_6 = Q_3 \cap Q_5 \in S_3$, $S_4 \to S_3$ the blowup of $q_6$, $Q_6$ its exceptional divisor. Let $q_7 = Q_6 \cap Q_3 \in S_4$ and $\tilde{S} = S_5 \to S_4$ the blowup of $q_7$. We denote by

$$\rho : \tilde{S} \to S_1 \quad and \quad \pi : \tilde{S} \to \mathbb{A}^2$$

the compositions of the blowups introduced above. As explained, each point $q_i$ defines a divisor $E_i \subset \tilde{S}$. If $(m_0, \dots, m_7) \in \mathbb{N}^8$, the ideal $\pi_*(\mathcal{O}_{\tilde{S}}(-\sum m_i E_i))$ is a punctual subscheme supported by $q_0$ which we will represent graphically with a label $m_i$ at the point of the Enriques diagram corresponding to $q_i$. For instance, the subscheme $\pi_*(\mathcal{O}_{\tilde{S}}(-8E_0 - 2E_1 - E_2 - E_4 - 3E_3))$ is associated with the following diagram.

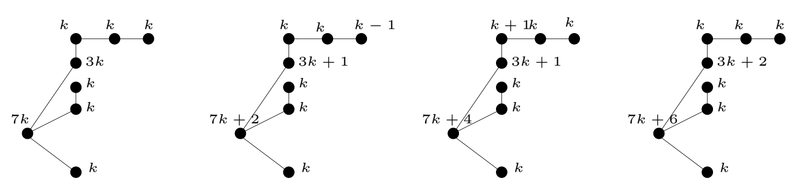



The following theorem describes the successive collision of four fat points which approach on curves $C_i$ with distinct tangent directions. This includes in particular the generic successive collision.

**Theorem 19.** *Let $q_0 \in \mathbb{A}^2$, $q_1, q_2, q_3$ three distinct tangent directions at $q_0$ and $C_1, C_2, C_3$ be three smooth curves passing through $p_0$ with tangent direction $q_1, q_2, q_3$. Let $Z_{mmmm}$ be the collision of the fat points $p_0^m, p_1^m, p_2^m, p_3^m$ where:*

- *$p_0$ is located at $q_0$,*
- *$p_1$ moves on the curve $C_1$ (resp. $p_2$ on $C_2$, $p_3$ on $C_3$).*

*Then $Z_{mmmm}$ is defined by the following Enriques diagram, which depends on $m$ modulo 4.*




$m = 4k$ $\quad$ $m = 4k+1$ $\quad$ $m = 4k+2$ $\quad$ $m = 4k+3$

*Proof.* All cases are similar and we prove the theorem in the case $m = 4k$. We choose a formal neighborhood $\xi$ of $p = (q_1, q_2, q_3) \in (S_1)^3$ such that $Q_0 \subset S_1$ is defined by the equation $x_1 = 0$ around each $q_i$ and such that $C_3$ is defined by $x_2 = 0$ around $q_3$ (this is possible since $C_3$ is smooth). Let

$n = (m-1, m-5, \ldots, 3)$. Let $F_m$ be the staircase defined by the height function $h_{F_m}(d) = h_{R_m}([\frac{d}{2}])$, and let $G = S(R_m, n)$ be the staircase obtained from $R_m$ by suppression of the slices indexed by $n$. Let $X_\xi(R_k, F_k, G_k) \subset S_1$ be the subscheme defined by the formal neighborhood $\xi$ and the staircases $R_k, F_k, G_k$. According to the correspondance between complete ideals and monomial subschemes formulated in [8], if the $m_i$'s are the integers defined in the Enriques diagram,

$$\rho_* \mathcal{O}_{\tilde{S}}(-\sum m_i E_i)) = \mathcal{O}_{S_1}(-m_0 Q_0 - X_\xi(R_k, F_k, G))) \quad (*)$$

Let $J(p_3)$ denote the ideal of $Z_{mmm} \cup p_3^m$. I claim that we are done if we prove the inclusion

$$\lim_{p_3 \to p_0} \eta^* J(p_3) \subset H^0(\mathcal{O}_{S_1}(-m_0 Q_0 - X_\xi(R_k, F_k, G)) \quad (**).$$

Indeed, we would then have the inclusions

$$\begin{array}{lcl}
I_{Z_{mmmm}} & \subset & \eta_* \eta^* I_{Z_{mmmm}} = \eta_* \eta^* \lim\limits_{p_3 \to p_0} J(p_3) \\
 & \subset & \eta_* \lim\limits_{p_3 \to p_0} \eta^* J(p_3) \\
 & \subset & \eta_* H^0(\mathcal{O}_{S_1}(-7kQ_0 - X_\varphi(R_k, F_k, G)) \;\; \textit{by } (**) \\
 & \subset & H^0(\eta_*(\mathcal{O}_{S_1}(-7kQ_0 - X_\varphi(R_k, F_k, G)) \\
 & \subset & H^0(\eta_* \rho_* \mathcal{O}_{\tilde{S}}(-\sum m_i E_i)) \;\; \textit{by } (*) \\
 & \subset & I_Z \text{ where } I_Z = \pi_* \mathcal{O}_{\tilde{S}}(-\sum m_i E_i).
\end{array}$$

According to [2], since the Enriques diagram defining $Z$ is unloaded, $deg(Z) = \sum \frac{m_i(m_i+1)}{2}$ which is immediatly checked to be $4\frac{4k(4k+1)}{2} = deg(Z_{mmmm})$. Summing up, $Z$ and $Z_{mmmm}$ are two punctual subschemes of the same degree with $I_{Z_{mmmm}} \subset I_Z$, thus they are equal.
It remains to prove the displayed inclusion $(**)$ using our theorem. By [3] or [11],

$$\eta^* I_{Z_{mmm}} = H^0 \mathcal{O}_{S_1}(-6kQ_0 - X_\psi(R_{2k}, F_{2k}))$$

where $\psi$ is the formal neighborhood of $(q_1, q_2)$ induced by the formal neighborhood $\xi$ of $(q_1, q_2, q_3)$. Thus

$$\lim_{p_3 \to p_0} \eta^* J(p_3) = \lim_{t \to 0} \mathcal{L}(-X_\varphi(R_m, t, v = 1))$$

where $\varphi$ is the formal neighborhood of $p_3$ induced by the formal neighborhood $\xi$ of $(q_1, q_2, q_3)$ and $\mathcal{L} = H^0(\mathcal{O}_{S_1}(-6kQ_0 - X_\psi(R_{2k}, F_{2k})))$. To apply theorem 1 with $X = S_1$, $s = 1$,$D = Q_0$, and $n = (m-1, m-5, \ldots, 3)$, the verification $\mathcal{L}((-i+1)D - Z_i) = \mathcal{L}(-iD)$ is needed. Elements of $\mathcal{L}((-i+1)D - Z_i)$ are sections of $\mathcal{O}_{S_1}((-6k - i + 1)Q_0)$ that vanish on

$$X_\psi(R_{2k-i+1}, F_{2k-i+1}) \cup Z_i = X_\xi(R_{2k-i+1}, F_{2k-i+1}, T(R_m, m-1-4(i-1))).$$

Since the intersection

$$Q_0 \cap X_\xi(R_{2k-i+1}, F_{2k-i+1}, T(R_m, m-1-4(i-1)))$$

has degree $3(2k-i+1)+(4i-3)$ greater than the degre $6k+i-1$ of the restriction $\mathcal{O}_S((-6k-i+1)Q_0)_{|Q_0}$, it follows that any section of $\mathcal{L}((-i+1)D-Z_i)$ vanishes on $D$. Thus we can apply the theorem and we get:

$$\begin{aligned}
\lim_{t\to 0} \mathcal{L}(-X_\varphi(R_m,t,1)) \quad \subset \quad & \mathcal{L}(-kQ_0 - X_\varphi(S(R_m,n))) \\
& = \\
& H^0(\mathcal{O}_{S_1}(-7kQ_0 - X_\psi(R_k,F_k) - X_\varphi(S(R_m,n)))) \\
& = \\
& H^0(\mathcal{O}_{S_1}(-m_0Q_0 - X_\xi(R_k,F_k,S(R_m,n))),
\end{aligned}$$

which concludes the proof. ■

# References


[1] J. Alexander and A. Hirschowitz. An asymptotic vanishing theorem for generic unions of multiple points. *Invent. Math.*, 140(2):303–325, 2000.

[2] E. Casas-Alvero. Infinitely near imposed singularities and singularities of polar curves. *Math. Ann.*, 287(3):429–454, 1990.

[3] L Evain. *Collisions de trois gros points sur une surface algébrique*. PhD thesis, PhD., Nice, 1997.

[4] Laurent Evain. Calculs de dimensions de systèmes linéaires de courbes planes par collisions de gros points. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(12):1305–1308, 1997.

[5] A. Grothendieck. Éléments de géométrie algébrique. I. Le langage des schémas. *Inst. Hautes Études Sci. Publ. Math.*, (4):228, 1960.

[6] Alexander Grothendieck. Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert. In *Séminaire Bourbaki, Vol. 6*, pages Exp. No. 221, 249–276. Soc. Math. France, Paris, 1995.

[7] A Hirschowitz. La méthode d'horace pour l'interpolation à plusieurs variables. *Manuscripta Mathematica*, 50:337–388, 1985.

[8] Evain Laurent. La fonction de Hilbert de la réunion de $4^h$ gros points génériques de $\mathbf{P}^2$ de même multiplicité. *J. Algebraic Geom.*, 8(4):787–796, 1999.

[9] M Nagata. On rational surfaces, II. *Memoirs of the College of Science, University of Kyoto*, XXXIII(2):271–293, 1960.

[10] Masayoshi Nagata. On the 14-th problem of Hilbert. *Amer. J. Math.*, 81:766–772, 1959.

[11] Charles Walter. Collisions of three fat points on an algebraic surface. *Prépublication 412, Univ. Nice*, pages 1–7, 1995.